Approaches for Angle of Arrival Estimation

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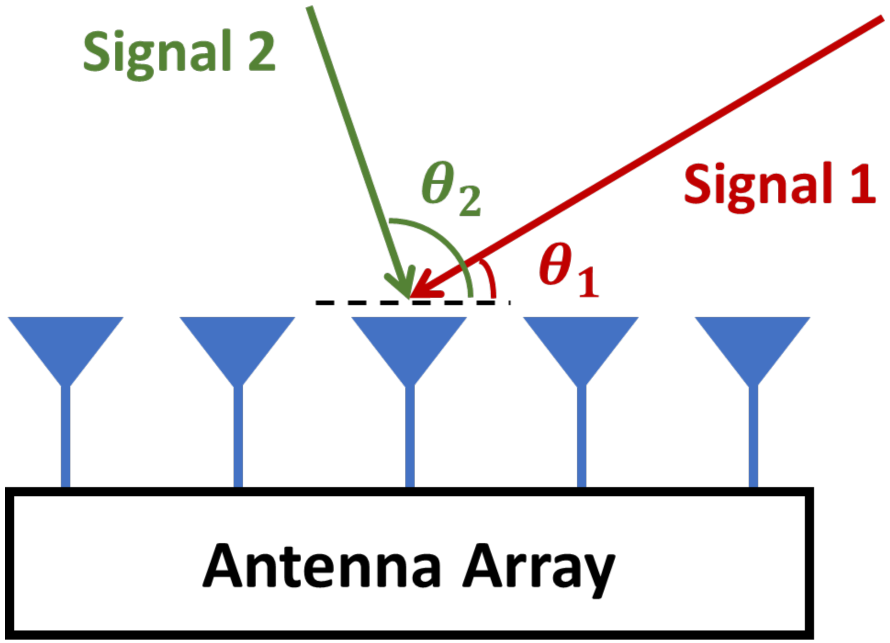
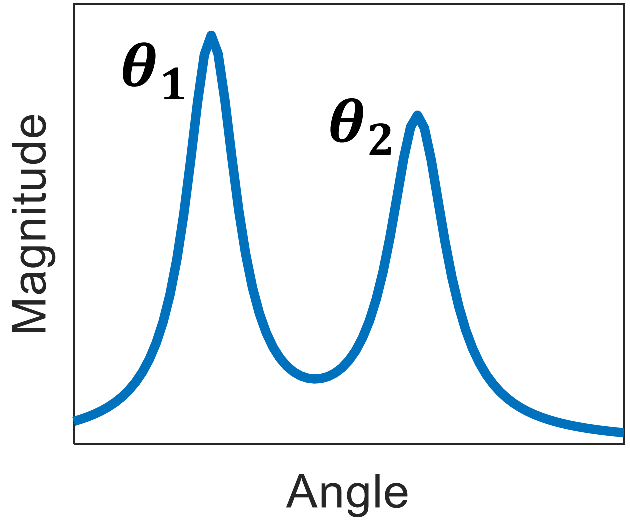
Angle of Arrival (AoA)

- Definition: the elevation and azimuth angle of incoming signals
- Also called direction of arrival (DoA)



AoA Estimation

- Applications: localization, tracking, gesture recognition, ……
- Requirements: antenna array
- Approaches:
  - Generate a power profile over various incoming angles
  - Determine all AoA θ_i



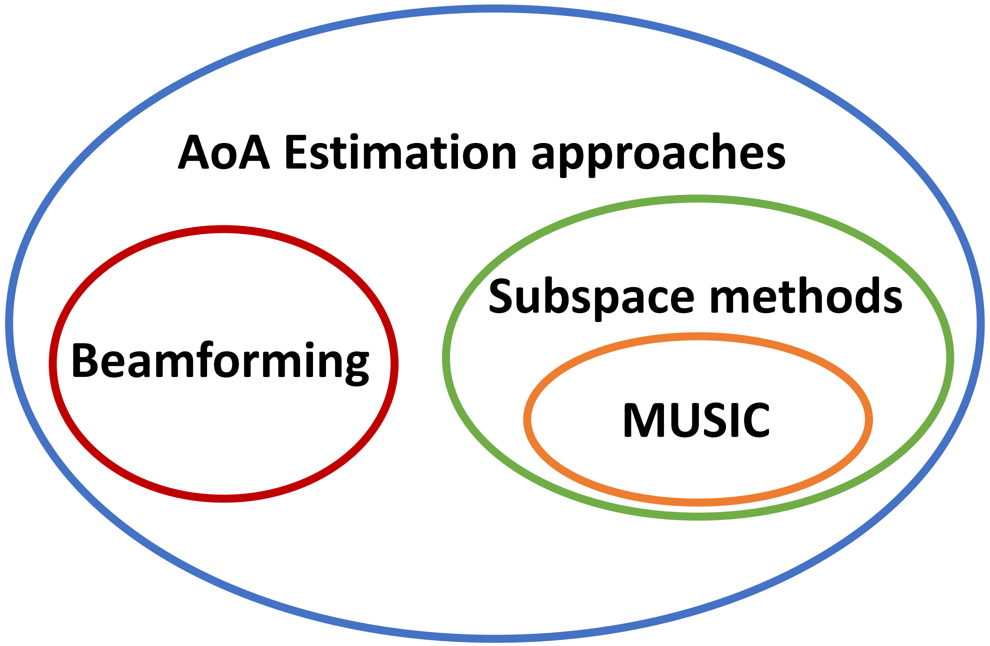
Related Concepts

- Synthetic aperture radar (SAR)
  - Using a moving antenna to emulate an array
  - Alternative way of using physical antenna array
  - **NOT** an estimation approach in the context of AoA
  - Most AoA estimation methods can be applied to both physical antenna array and SAR

- In this presentation, we only focus on antenna array
  - May require some modification when applied to SAR

Related Concepts

- Beamforming
  - A class of AoA estimation approaches
- MUSIC
  - A specific algorithm in subspace-based approaches



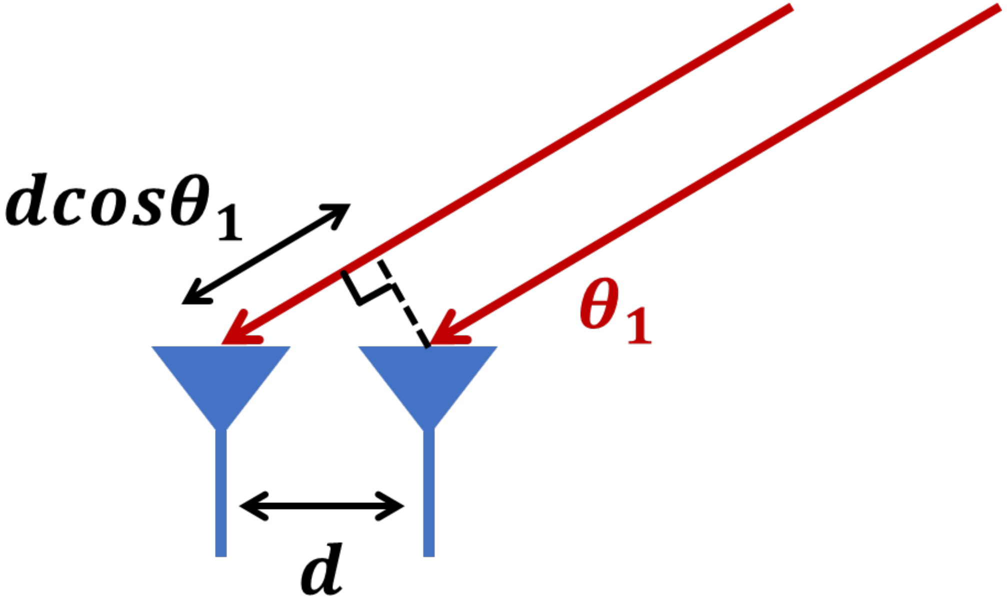
Approaches for AoA Estimation

- **Naïve approach**
- Beamforming approaches
  - **Bartlett method**
  - **MVDR**
  - Linear prediction
- Subspace based approaches
  - **MUSIC** and its variants
  - ESPIRIT
- Maximum likelihood estimator
- ……

Key Insights

- Phase changes over antennas are determined by the incoming angle
- Far-field assumption
- Phase of the antenna 1: ϕ_1
- Phase of the antenna 2: ϕ_1
- Then the difference is given by

$$\boldsymbol{\phi_2 - \phi_1 = 2\pi \frac{dcos\theta_1}{\lambda} + 2k\pi}$$



Naïve approach

- Determine AoA based on the phase difference of two antenna

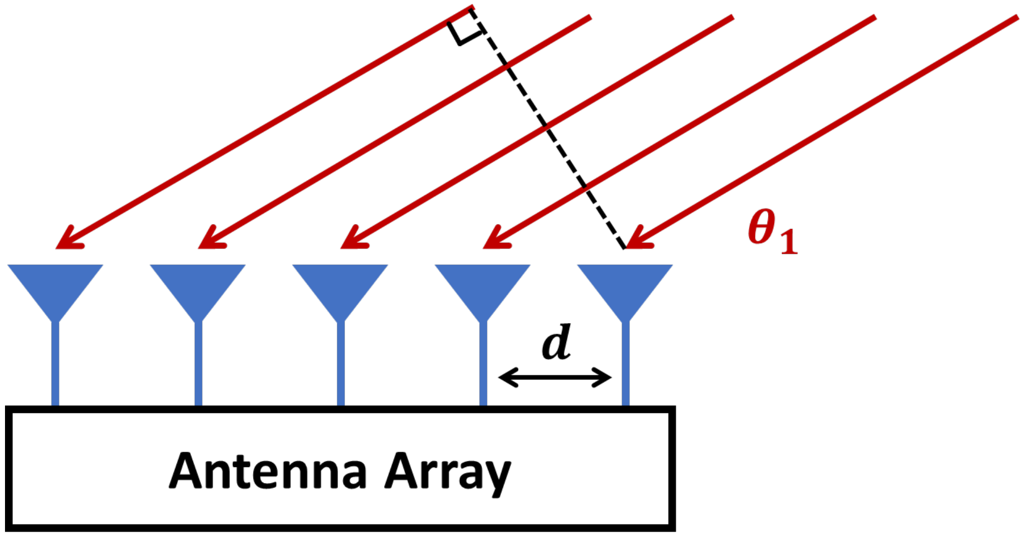
$$\cos\theta_1 = \left(\frac{\Delta\phi}{2\pi} - k\right)\frac{\lambda}{d}$$

- Problems:
  - Works for only one incoming signals
  - Phase measurement could be noisy
  - Ambiguity
- Adopted and improved by **RF-IDraw**

Using Antenna Array

- Received signals at m-th antenna:

$$\boldsymbol{x_m(t)} = \sum_{n=1}^{N} \boldsymbol{s_n(t)\, e^{j\cdot 2\pi\cdot \tau_n \cdot (m-1)}} + \boldsymbol{n_m(t)}$$



$s_n(t)$: n-th source signals

$\tau_n = \frac{d\cos\theta_n}{\lambda}$: phase shift per antenna

N : the number of sources

M : the number of antennas

$n_m(t)$: noise terms

Using Antenna Array

- Matrix form:

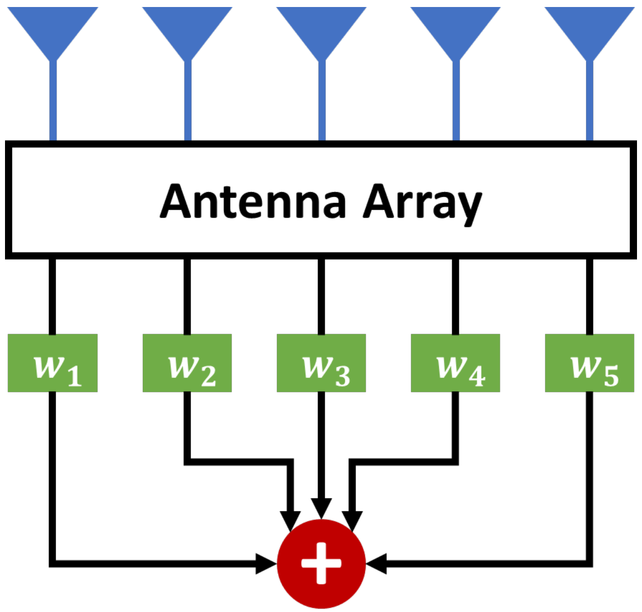
$$\begin{bmatrix} \boldsymbol{x_1(t)^T} \\ \boldsymbol{x_2(t)^T} \\ \vdots \\ \boldsymbol{x_M(t)^T} \end{bmatrix} = [\boldsymbol{a(\theta_1)} \quad \boldsymbol{a(\theta_2)} \quad \cdots \quad \boldsymbol{a(\theta_N)}] \begin{bmatrix} \boldsymbol{s_1(t)^T} \\ \boldsymbol{s_2(t)^T} \\ \vdots \\ \boldsymbol{s_N(t)^T} \end{bmatrix} + \begin{bmatrix} \boldsymbol{n_1(t)^T} \\ \boldsymbol{n_2(t)^T} \\ \vdots \\ \boldsymbol{n_M(t)^T} \end{bmatrix}$$

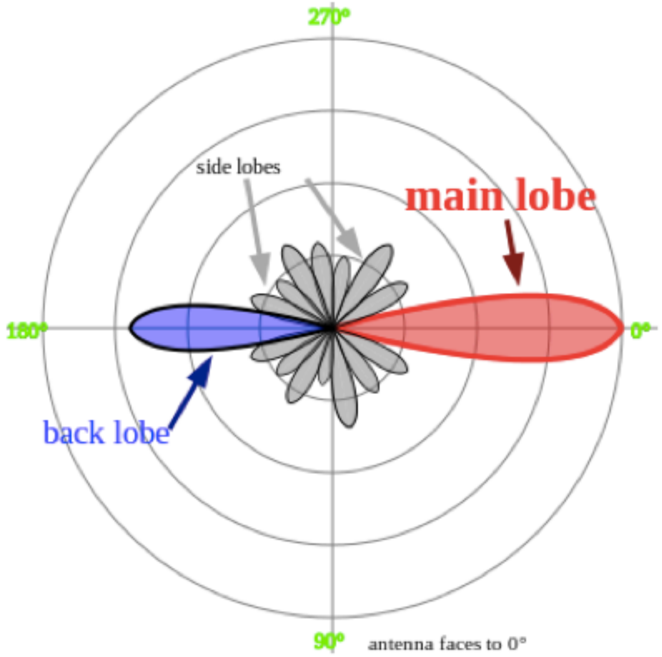
$$\boldsymbol{X = AS + N}$$

Steering vector: $\boldsymbol{a(\theta) = \left[1 \;\; e^{j2\pi\tau(\theta)} \;\; e^{j2\pi\tau(\theta)\cdot 2} \;\; \ldots \;\; e^{j2\pi\tau(\theta)(M-1)}\right]^T}$

Beamforming at the Receiver

- Definition: a method to create certain **radiation pattern** by combining signals from different antennas with different **weights**.
- Will magnify the signals from certain direction while suppressing those from other directions





Beamforming at the Receiver

- Signals after beamforming using a **weight vector $\boldsymbol{w}$**

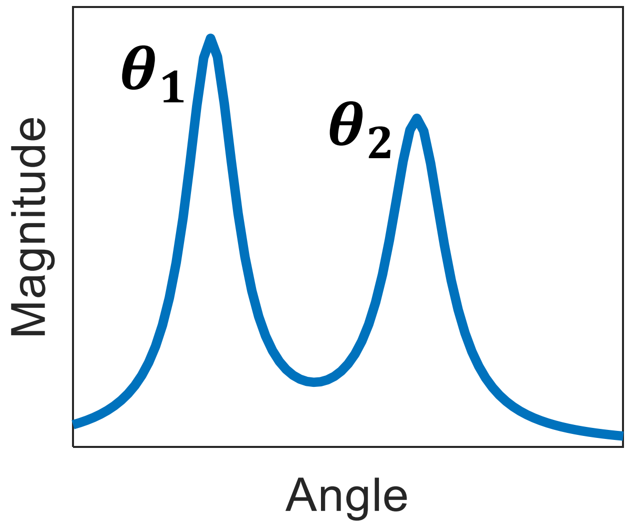
$$\boldsymbol{Y} = \boldsymbol{w}^H \boldsymbol{X}$$

- By selecting different w, the received signal Y will contain the signal sources arrived from different direction.

- Beamforming techniques are widely used in wireless communications

Beamforming at the Receiver

- Adjust the weight vector to rotate the radiation pattern to angle θ
- Measure the received signal strength $P(\theta)$
- Repeat this process for any θ in [0, pi]
- Plot $(\theta, P(\theta))$
- Peaks in the plot indicates the angle of arrival



Bartlett Beamforming

- Also called: correlation beamforming, conventional beamforming, delay-and-sum beamforming, or Fourier beamforming
- **Key idea**: magnify the signals from certain direction by compensating the phase shift
- Consider one source signal $s(t)$ arrived at angle θ_0
- Signal at m-th antenna: $\mathrm{x_m(t)} = s(t) \cdot e^{j \cdot 2\pi \cdot \tau(\theta_0)(m-1)}$
- Weight at m-th antenna: $\mathrm{w_m} = e^{j \cdot 2\pi \cdot \tau(\theta)(m-1)}$
- Only when $\theta = \theta_0$, the received signal $\mathrm{Y} = \mathrm{w^H X} = \sum w_m^* x_m(t)^T$ is maximized

Phase shift

Bartlett Beamforming

- Weight vector for beamforming angle θ:

$$\boldsymbol{w} = \boldsymbol{a}(\boldsymbol{\theta})$$

This is why it is called steering vector

- Signal power at angle θ:

$$\boldsymbol{P}(\boldsymbol{\theta}) = \boldsymbol{Y}\boldsymbol{Y}^{\boldsymbol{H}} = (\mathbf{w}^{\boldsymbol{H}}\boldsymbol{X})(\mathbf{w}^{\boldsymbol{H}}\boldsymbol{X})^{\boldsymbol{H}} = \mathbf{w}^{\boldsymbol{H}}\boldsymbol{X}\boldsymbol{X}^{\boldsymbol{H}}\boldsymbol{w} = \mathbf{w}^{\boldsymbol{H}}\boldsymbol{R}_{\boldsymbol{X}\boldsymbol{X}}\boldsymbol{w} = \boldsymbol{a}^{\boldsymbol{H}}(\boldsymbol{\theta})\boldsymbol{R}_{\boldsymbol{X}\boldsymbol{X}}\boldsymbol{a}(\boldsymbol{\theta})$$

Covariance matrix

- Used by **Ubicarse** with SAR

Bartlett Beamforming

- Works well when there is only one source signal
- Suffers when there are multiple sources: **very low resolution**

Minimum Variance Distortionless Response (MVDR)

- Also called Capon's beamforming
- **Key idea**: maintain the signal from the desired direction while minimizing the signals from other direction
- Mathematically, we want to find such weight vector $\boldsymbol{w}$ for the beaming angle θ

$$\min\{\boldsymbol{Y}\boldsymbol{Y}^H\} = \min\{\boldsymbol{w}^H \boldsymbol{R}_{XX} \boldsymbol{w}\}$$

$$\text{s.t.} \quad \boldsymbol{w}^H(\boldsymbol{a}(\boldsymbol{\theta})\boldsymbol{s}(\boldsymbol{t})^T) = \boldsymbol{s}(\boldsymbol{t})^T \rightarrow \boldsymbol{w}^H\boldsymbol{a}(\boldsymbol{\theta}) = \mathbf{1}$$

Maintain the signals from angle $\boldsymbol{\theta}$

MVDR

- Weight vector for beamforming angle θ:

$$\boldsymbol{w} = \frac{\boldsymbol{R}_{\boldsymbol{XX}}^{-1}\boldsymbol{a}^{\boldsymbol{H}}(\boldsymbol{\theta})}{\boldsymbol{a}(\boldsymbol{\theta})\boldsymbol{R}_{\boldsymbol{XX}}^{-1}\boldsymbol{a}^{\boldsymbol{H}}(\boldsymbol{\theta})}$$

- Signal power at angle θ:

$$\boldsymbol{P}(\boldsymbol{\theta}) = \boldsymbol{Y}\boldsymbol{Y}^{\boldsymbol{H}} = \boldsymbol{w}^{\boldsymbol{H}}\boldsymbol{R}_{\boldsymbol{XX}}\boldsymbol{w} = \frac{\boldsymbol{1}}{\boldsymbol{a}(\boldsymbol{\theta})\boldsymbol{R}_{\boldsymbol{XX}}^{-1}\boldsymbol{a}^{\boldsymbol{H}}(\boldsymbol{\theta})}$$

MVDR

- Resolution is significantly enhanced compared to Bartlett method
- But still not good enough
- Better beamforming approaches are developed, e.g., Linear Prediction
- Or resort to **subspace based approaches**

Subspace Based Approaches

- Beamforming is a way of shaping received signals
  - Can be used for estimating AoA
  - Can also be used for directional communications
- Subspace based approaches are specially designed for parameter (i.e., AoA) estimation using received signals
  - Cannot be used for extracting signals arrived from certain direction
- Subspace based approaches **decompose** the received signals into "signal subspace" and "noise subspace"
  - Leverage special properties of these subspaces for estimating AoA

Multiple Signal Classification (MUSIC)

- Key ideas: we want to find a vector q and a vector function $f(\theta)$
- Such that $\boldsymbol{q^H f(\theta) = 0}$ if and only if $\theta = \theta_i$ (i.e., one of AoA)
- Then we can plot $\boldsymbol{p(\theta)} = \frac{1}{\|q^H f(\theta)\|^2} = \frac{1}{f^H(\theta) q q^H f(\theta)}$
- The peaks in the plot indicates AoA
- We can expect **very sharp peak** since $q^H f(\theta) = 0$, so the inverse of its magnitude is infinity

How to find q and $f(\theta)$

Multiple Signal Classification (MUSIC)

- MUSIC gives a way to find a pair of q and $f(\theta)$
- The signals from antenna array

$$\boldsymbol{X} = \boldsymbol{AS} + \boldsymbol{N}$$

- Covariance matrix of the signals

$$\boldsymbol{R_{XX}} = \boldsymbol{E}[\boldsymbol{XX^H}] = \boldsymbol{E}[\boldsymbol{ASS^H A^H}] + \boldsymbol{E}[\boldsymbol{NN^H}]$$

$$\boldsymbol{R_{XX}} = \boldsymbol{AE}[\boldsymbol{SS^H}]\boldsymbol{A^H} + \boldsymbol{\sigma^2 I}$$

$$\boldsymbol{R_{XX}} = \boldsymbol{AR_{SS}A^H} + \boldsymbol{\sigma^2 I}$$

Noise terms

Signal terms

MUSIC

- Consider the signal term
- R_{SS} is $N \times N$ matrix, where N is the number of source signals
  - R_{ss} has the rank equal to N if source signals are independent
- A is $M \times N$ matrix, where M is the number of antenna
  - A has full column rank
  - The signal term is $M \times M$ matrix, and its rank is N
  - The signal term has N positive eigenvalues and $M - N$ zero eigenvalues, if M>N
  - There are $M - N$ eigenvectors q_i such that $A R_{SS} A^H q_i = 0$
  - Then $A^H q_i = 0$, where $A = [a(\theta_1) \quad a(\theta_2) \quad \cdots \quad a(\theta_N)]$
  - Then $q_i^H a(\theta) = 0$ if $\theta = \theta_i$ ← **What we want !!!**

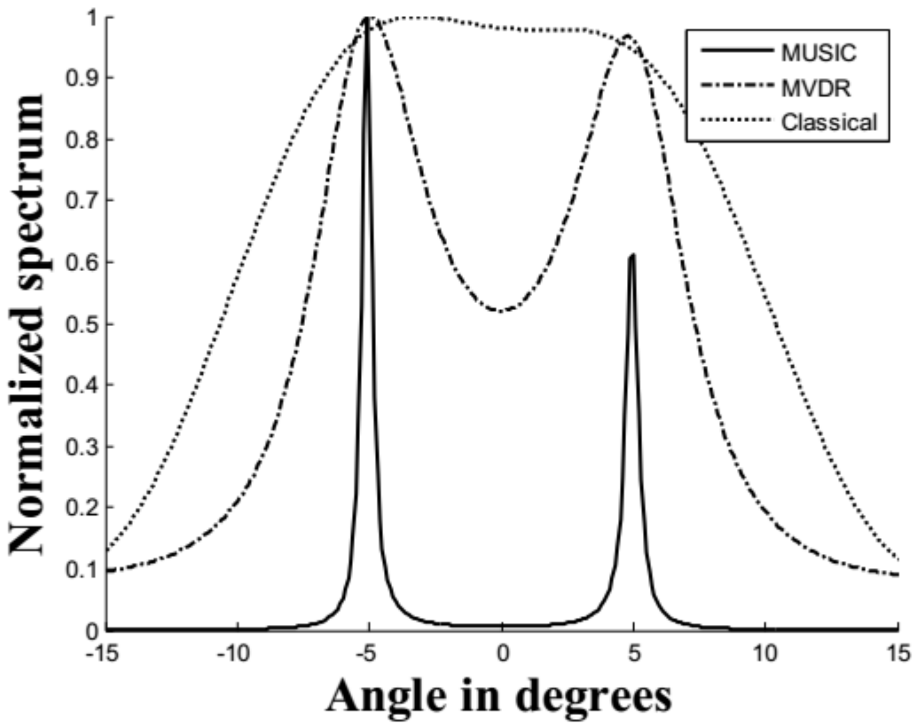
MUSIC

- $a(\theta)$ is the steering function, so it is known
- Needs to determine q_i, which needs the eigenvalue decomposition of the signal term.
- We don't know the signal term; we only know the sum of the signal term and the noise term, i.e., R_{XX}
- All of eigenvectors of the signal term are also ones for R_{XX}, and corresponding eigenvalues are added by σ^2
- Only need to find the eigenvectors of R_{XX} with eigenvalues equal to σ^2

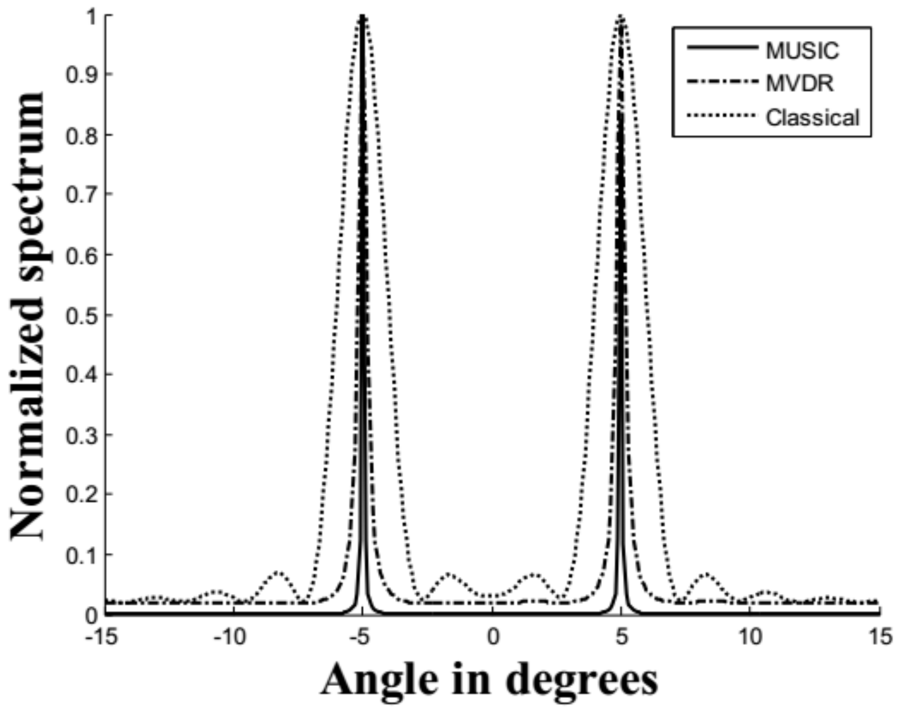
MUSIC

- Derive R_{XX}
- Perform eigenvalue decomposition on R_{XX}
- Sort eigenvectors according to their eigenvalues in descent order
- Select last $M - N$ eigenvectors q_i
- **Noise space matrix** $Q_N = [q_{M+1} \;\; q_{M+2} \;\; \dots \;\; q_N]$
- $Q_N^H a(\theta) = 0$ for any AoA θ_i
- Plot $p(\theta) = \frac{1}{a^H(\theta) Q_N Q_N^H a(\theta)}$ and find the peaks

Performance Comparison

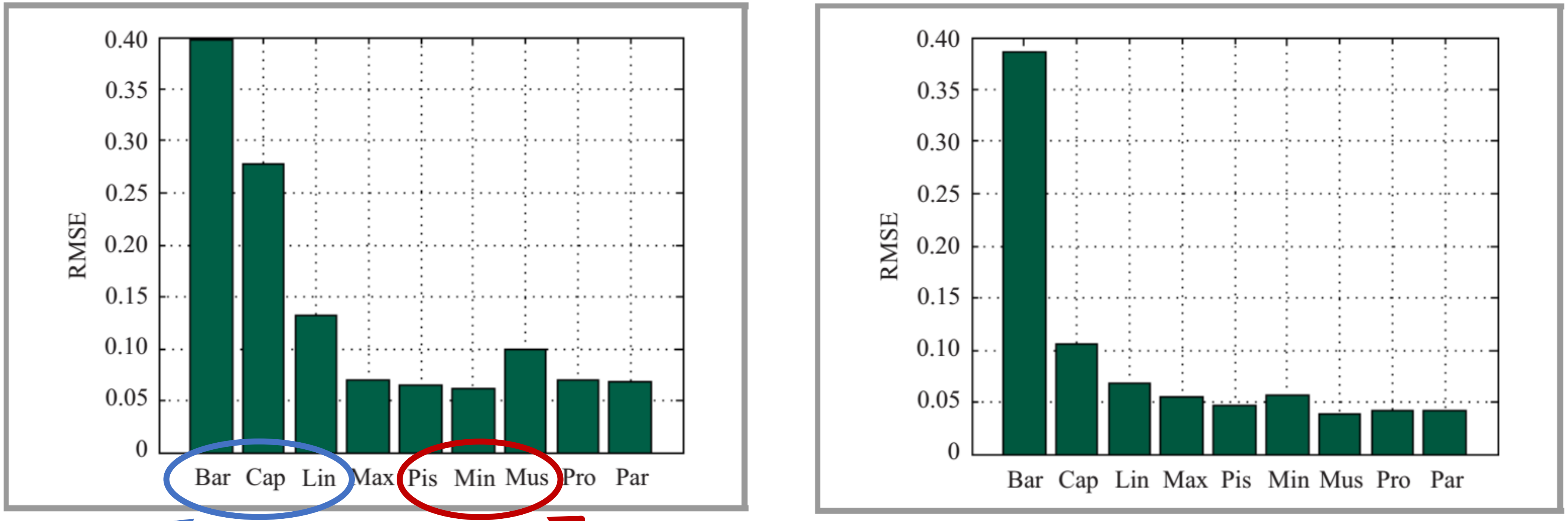


(a) 10 antennas



(a) 50 antennas

Performance Comparison



(a) SNR 1dB

(b) SNR 20dB

Beamforming approaches

Music variants